THE SPONTANEOUS MAGNETIZATION OF TWO-DIMENSIONAL RECTANGULAR ISING MODEL: EFFECTIVE-FIELD THEORY ANALYSIS

Antonia C. Paz¹, Ingrid N. da Costa² and Minos A. Neto³

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Resumo
Neste artigo de revisão estudaremos o modelo de Ising bidimensional. O modelo consiste de interações ferromagnéticas $J_y = \lambda J_x$ nas direções $x(y)$. Para alguns valores do parâmetro $\lambda$ serão obtidos a magnetização como uma função da temperatura $T$, usando da técnica do operador diferencial, proposta por Honmura e Kaneyoshi (1979), baseada na teoria de campo efetivo com aglomerado finito de $N = 1$ spins (EFT-1).

Palavras-Chave: Modelo de Ising, Teoria de campo efetivo, rede quadrada.

Abstract
In this paper we have studied the two-dimensional rectangular Ising model. The model consists of ferromagnetic interactions $J_y = \lambda J_x$ in the $x(y)$ direction. For some values $\lambda$ we obtain the magnetization $m$ as a function temperature $T$, using the framework of the differential operator technique, proposal by Honmura and Kaneyoshi (1979), based in the effective field theory with finite cluster of $N = 1$ spins (EFT-1).

Key-words: Ising model, effective-field theory, square lattice.

¹ Universidade Federal do Amazonas, Departamento de Física, 3000, Japiim, 69077-000, Manaus-AM, Brazil.
² Universidade Federal do Amazonas, Instituto de Computação - IComp, 3000, Japiim, 69077-000, Manaus-AM, Brazil.
³ Universidade Federal do Amazonas, Departamento de Física, 3000, Japiim, 69077-000, Manaus-AM, Brazil.
1. Introduction

The Ising model is a well-known and studied model in the statistical mechanics. Because of its simplicity, this model has attracted the concerted attention for over 80 years. The Ising model was first solved in one-dimensional, where was observed that it have not phase transition at a finite temperature (i.e., $T_c = 0$ (ISING, 1925). In 1944, Onsager (1944) obtained an exact expression for the free energy of the Ising model on a square lattice in zero field, and in 1952 Yang (1952) presented a computation of the spontaneous magnetization. Recently, Zhang (2007) has presented a conjectured expression for the free energy and spontaneous magnetization of the 3d Ising model, but some authors have argued that these conjectures are false (WU et al., 2008). The most reliable estimates for the phase transition temperature $T_c$ in the 3d Ising model were computed by using high-temperature series and Monte Carlo (MC) methods (MCCOY e WU, 1973; BAXTER, 1982). The best estimates rely on finite-size scaling of MC simulations on a simple cubic lattice is $k_B T_c/J = 4.5115240(21)$, while on a square lattice we have an exact value $k_B T_c/J = 2/\ln (1 + \sqrt{2}) \approx 2.269185314$....

Another motivation to study the Ising model is because it can be used to describe the critical behavior of a broad class of materials, including easyaxis magnets, binary alloys, simple liquids and their mixture, polymer solutions, subnuclear matter, etc. (ANDREI N. et al., 1983; CHUNG et al., 1983; WHITE, 1993).

It is the purpose of the present paper to calculate the spontaneous magnetization (i.e., the intensity of magnetization at zero external magnetic field) of a two-dimensional Ising model of a ferromagnet. Van der Waerden (1939) and Ashkin and Lamb had obtained a series expansion of the spontaneous magnetization that converges very rapidly at low temperature (KUZ'MIN, 2005). Near the critical temperature, however, their series expansion cannot be used. We shall here obtain a close expression for the spontaneous magnetization by the matrix method which was introduced into the problem of the statistical of a two-dimensional Ising model by Montroll and Kramers and Wannier (HONMURA e KANEYOSHI, 1979; TUCKER, 1994). Onsager gave in 1944 a complete solution of the matrix problem. His method was subsequently greatly simplified by B. Kaufman (1993), and the result has been used to calculate the short-range order in the crystal lattice (FISHMAN e VIGNALE, 1991).

This paper will be presented in the following way: in the Section 2, we will develop with details the model of Montroll being used effective-field theory in clusters with one spin (EFT-1) (STRIEB e CALLEN, 1963; FISHMAN e LIU, 1992; IDOGAKI e URYÜ, 1992; JIANG e FISHMAN, 1993; CHAKRABORTY, 1993; DO NASCIMENTO et al., 2012), in the Section 3 the behavior of the magnetization $m$ as a function reduced temperature given by $T$, varying the exchange ratio $J_y/J_x$ among the directions $x$ and $y$, and finally in the Section 4 the due conclusions of this paper.

2. Model and formalism

A. Hamiltonian

The model to be studied is the nearest-neighbor ($nn$) Ising antiferromagnetic in a longitudinal magnetic field, which is described by the following Hamiltonian:

$$H = -J_x \sum_{i,\delta_x} \sigma^x_i \sigma^x_{i+\delta_x} - J_y \sum_{i,\delta_y} \sigma^y_i \sigma^y_{i+\delta_y}$$  

(1)

where $\sigma^\mu_i$ is the ($\mu = x, y, z$) component spin-1/2 Pauli operator at site $i$, $J_x (J_y)$ is the exchange coupling along the $x(y)$ axis, $\delta_x (\delta_y)$ denotes the nearest-neighbor vector along the $x(y)$ axis and we define the parameter $\lambda = J_y/J_x$. 
On the other hand, in the case of
\( J \) 
(y > 0) we have the SAF (see Figure 1a) and
\( J_x < 0 \)
AF (see figure 1b) states, respectively. The
ground-state of the model (1) is characterized
by a parallel spin orientation in the horizontal
and vertical direction, see Figure 1c. In the
absence of magnetic field \( H \), the criticality
of the three magnetic states (F, AF and SAF)
are equivalent, i.e., \( T_C(0) = T_N(0) \). In the
presence of the field the F state present not
phase transition, while that the AF and SAF
states have field induced phase transition with
\( T_N^{SAF}(H) \leq T_N^{AF}(H) \) (WEISS, 1907).

![Figure 1: Representations of the SAF (a), AF (b)
and F (c) ground states.](image)

The model is exactly solved for \( H = 0 \),
because the lattice is now composed of
independent planes, so the critical temperature
may be calculated by using the relation given
in Onsager (1944)

\[
\sinh \left( \frac{2J_x}{k_B T_N} \right) \sinh \left( \frac{2J_y}{k_B T_N} \right) = 1(2)
\]

where for the particular isotropic case \( J_x = J_y = J(\lambda = 1) \) we have \( k_B T_N / J = 2 / \ln \left( 1 + \sqrt{2} \right) \). Effective-Field Theory To
begin with an effective-field treatment, we need the averages of a general function
involving spin operator components \( O(\{n\}) \),
these are obtained by effectuating the
following operations (WEISS, 1907)

\[
\langle \{n\} \rangle = \frac{(TR_{n} O(\{n\}) e^{-\beta H_n})}{TR_{n} e^{-\beta H_n}} (3)
\]

where the partial trace \( TR_{n} \) is taken over the set
\( n \) of spin variables (finite cluster) specified by
the multi-site spin Hamiltonian \( H_n \) and \( \langle \cdots \rangle \)
indicates the usual canonical thermal average.

The method deals with the effects of the
surrounding spins of a finite cluster with \( N \)
spins through a convenient differential-
operator technique in such a way that all
relevant selfspin correlations are considered
(BETHE, 1935). In contrast, the spin
correlations are neglected in the mean-field
procedure. The interactions within the cluster
are exactly treated and the effect of the
remaining lattice spins is dealt by using the
random phase approximation (RPA).

In order to treat the model (1) by the
EFT approach, we consider a simple cluster on
a lattice consisting of a central spin and \( z \)
perimeter spins being the nearest-neighbors
of the central one. The nearest-neighbor spins are
substituted by an effective field produced by
the other spins, which can be determined by
the condition that the thermal average of the
central spin is equal to that of its nearest-
neighbor ones. The Hamiltonian for this
cluster is given by

\[
\mathcal{H}_1 = \left( -J_x \sum_{\bar{\delta}_x} \sigma_{(i+\bar{\delta}_x)}^z - J_y \sum_{\bar{\delta}_y} \sigma_{(i+\bar{\delta}_y)}^z \right) \sigma_1^z. \tag{4}
\]

Using the Hamiltonian (4) in the
approximate Callen-Suzuki relation we obtain
the average magnetization as: \( m = \langle \sigma_1^z \rangle \) is
given by

\[
m = \langle \tanh(\alpha + \alpha_2) \rangle \tag{5}
\]

where

\[
\alpha_1 = J_x \sum_{\delta_x} \sigma_{(i+\delta_x)}^z \\
\alpha_2 = J_y \sum_{\delta_y} \sigma_{(i+\delta_y)}^z
\]

Now using the identity \( \exp(aD_x + bD_y)F(x, y) = F(x + y + a + b) \) (where
\(D_\mu = \frac{\partial}{\partial \mu}\) is the differential operator and the van der Waerden relation for the two-state spin system (KIKUCHI, 1951), i.e. \(\exp(a \sigma_z^2) = \cosh(a) + \sigma_z^2 \sinh(a)\), the Eq. (5) can be rewritten as

\[
m = \left< \prod_{x} (\alpha_x + \sigma_z^x \beta_x) \prod_{y} (\alpha_y + \sigma_z^y \beta_y) \right> F(x, y)|_{x=y=0}
\]

with

\[
F(x, y) = \tanh[\beta(x + y)],
\]

where \(\alpha_\mu = \cosh(J_\mu D_\mu)\) and \(\beta_\mu = \sinh(J_\mu D_\mu)\).

### 3. Results and discussion

In Figure 2, the magnetization curve \(m\) is presented as function of the reduced temperature for selected values of \(\lambda (= 1.0, 0.8, 0.6, 0.4\) and 0.2). The two-dimensional rectangular Ising model exhibits a phase transition, with the presence of a second-order transition at \(T = T_c(\lambda)\). The numerical determination of the phase boundary (second-order phase transition) is obtained by solving the Eq. (10). Near the critical temperature, the ferromagnetic order parameter for behaves as

\[
m \sim t^\beta \text{ and } \chi \sim t^{-\gamma} \text{ where } T - T_c \text{ and the critical exponents are classical (i.e., } \beta = 1/2 \text{ and } \gamma = 1).\]

![Figure 2: Dependence of the magnetization \(m\) as a function of the critical temperature, \(k_B T/J_x\), for the two-dimensional rectangular Ising model with several values of \(\lambda\).](image)

Experimentally, it is also noted that the molar specific heat at vanishes field has singular behavior. This quantity may diverge
of according with $c \sim V T - T_c^{-\alpha}$. An example of this behavior was obtained by Connelly et al. for Ni (YAMAMOTO, 2009). For a singularity classified by $\alpha=0$ the divergence is like-logarithmic. Some values of the critical exponents $\alpha$, $\beta$ and $\gamma$ were obtained experimentally as: Fe ($\alpha=-0.12$, $\beta=0.39$, $\gamma=1.34$) (ETXE BARRIA et al. 2004), Co (KATORI e SUZUKI, 1988) ($\alpha=...$, $\beta=0.38$, $\gamma=1.34$), EuO (CHUNG, 2006) ($\alpha=-0.04$, $\beta=0.37$, $\gamma=1.40$) and EuS (CHUNG, 1952) ($\alpha=-0.13$, $\beta=0.36$, $\gamma=1.39$).

4. Conclusions
We will discuss in this section the spontaneous magnetization as a function of the critical temperature, for the two-dimensional rectangular Ising model with several values of $\lambda$. The behavior of the critical temperature is increasing function of the $\lambda$ (see Figure 2) in the one-dimensional ($\lambda = 0$) limit we obtain $T_c = 0$. This result differs from the classical approach that is $k_B T_c / J_\chi = 2.0$.

For an Ising chain ($\lambda=0$), the correlation length at low temperature presents an exponential divergence (CONELLY, 1971) $\xi_T \approx e^{4A/T}$, and for the quasi-one-dimensional limit at $T = 0$ we expect the critical behavior $\xi_\lambda \approx \lambda^{-1/\phi}$ ($\phi$ is the crossover exponent). Therefore, comparing the two correlation lengths, i.e., $\xi_T = \xi_\lambda$ (HELLER, 1967), we can explain the logarithmic divergence of the inverse critical temperature given by $k_B T_c / J_\chi \approx A / \ln (1/\lambda)$. The classical approach presents the linear behavior $k_B T_c / J_\chi = 2 + 2\lambda$, which shows an incorrect result in the one-dimensional limit ($\lambda = 0$) $T_c \neq 0$ (KOUVEL, 1968).

Disclosure
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Appendix A
The coefficients $a_0$, $a_1$, $a_2$, $a_3$ and $a_4$ are given by

$$a_0 = \alpha_2^2 \beta_2^2,$$

$$a_1 = 2(\alpha_2^2 \alpha_2 \beta_2 + \alpha_3 \alpha_2^2 \beta_2),$$

$$a_2 = (\alpha_2^2 \beta_2^2 + 4 \alpha_3 \beta_2 \alpha_2 \beta_2 + \alpha_2^2 \beta_2^2),$$

$$a_3 = 2(\alpha_2 \beta_2 \beta_2 + \alpha_2 \beta_2 \beta_2),$$

and

$$a_4 = \beta_2^2 \beta_2^2.$$

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